

The volume of causal diamonds, asymptotically de Sitter space-times and irreversibility

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The volume of causal diamonds, asymptotically de Sitter space-times and irreversibility

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ABSTRACT: In this note we prove that the volume of a causal diamond associated with an inertial observer in asymptotically de Sitter 4-dimensional space-time is monotonically increasing function of cosmological time. The asymptotic value of the volume is that of in maximally symmetric de Sitter space-time. The monotonic property of the volume is checked in two cases: in vacuum and in the presence of a massless scalar field. In vacuum, the volume flow (with respect to cosmological time) asymptotically vanishes if and only if future space-like infinity is 3-manifold of constant curvature. The volume flow thus represents irreversibility of asymptotic evolution in spacetimes with positive cosmological constant.

KEYWORDS: AdS-CFT Correspondence, Differential and Algebraic Geometry, dS vacua in string theory, Classical Theories of Gravity.

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1. Introduction

In this paper we continue the study of geometry of causal diamonds initiated in [1] and [2]. The focus of the present study is the irreversible behavior of the volume of a causal diamond in space-times asymptotic to de Sitter space. The causal diamonds play an important role in various recent classical and quantum investigations of cosmologies with positive cosmological constant (see [3] and [4] and references therein). In fact, the diamond appears rather naturally as the region accessible for experiments made by a hypothetical observer moving along a time-like geodesic. Imagine an observer that makes experiments by sending the light rays and detecting the signals that come back and has only a finite duration time τ for his/her experiments. Then the region of space-time that can be probed by this type of experiments is exactly the causal diamond associated with the observer. Geometrically, as was demonstrated in [1] and [2], the volume of a causal diamond encodes information on the curvature of the space-time. Moreover, this information is inherently irreversible: the differences of the space-time geometry inside the diamond from de Sitter geometry inevitably disappear as cosmological time progresses.

There are several indications in the literature that the cosmological evolution with positive cosmological constant is irreversible. The most straightforward way is to associate¹

¹I thank G. Gibbons for suggesting this point.

this to the irreversible growth of the cosmological horizon [5]. The entropy associated to the horizon is then non-decreasing in agreement with the laws of thermodynamics.

In a different development one considers a dual holographic description of de Sitter space-time in terms of a conformal field theory (CFT) [6] defined on a space-like boundary of the space-time. From the point of view of quantum CFT it is natural to define a function (known as C-function) which changes monotonically along the RG trajectories. In the dual description such a C-function is defined in terms of the space-time metric and its derivatives. One then shows that, under suitable energy conditions one has to impose on matter fields, the C-function changes monotonically with cosmological time [6].

In the present paper we suggest yet another manifestation of the irreversible cosmological evolution. We show that the volume of a causal diamond grows monotonically with cosmological time. The maximal value it approaches at future space-like infinity is that of volume in maximally symmetric de Sitter space-time.

2. The result

We consider an observer that follows a timelike geodesic γ in metric which is not exactly but only asymptotically de-Sitter, in the limit that his/her own proper time $t_q \rightarrow \infty$. We shall study the volume $V(\tau, t_q)$ of the causal diamond $\dot{I}^+(p) \cap \dot{I}^-(q)$ where p and q lie on γ in the limit when both $t_p, t_q \rightarrow \infty$ while $\tau = t_q - t_p$ is kept fixed. Thus both points p and q tend to future spacelike infinity \mathcal{I}^+ while the duration of the diamond τ is kept fixed. The entire diamond is in the asymptotic region and the volume of the diamond depends on the asymptotic geometry. In the limit when $t_q \rightarrow \infty$ the point q on the geodesic γ approaches the point q^+ of intersection of geodesic γ and the future space-like infinity \mathcal{I}^+ . The asymptotic metric in 4d geodesic coordinates takes the form

$$ds^2 = -dt^2 + e^{2t} g_{ij}^{(0)} dx^i dx^j \quad , \quad (2.1)$$

where t is the cosmological time and $g_{ij}^{(0)}(x)$ is an *arbitrary* 3d metric defined on future infinity of the 4-dimensional asymptotically de Sitter space-time. We skip the subleading terms, defined as a series in powers of e^{-t} , in (2.1). Throughout the paper we set the de Sitter radius $l = 1$.

A special role is played by the maximally symmetric de Sitter space-time.² This space-time is characterized by the fact that, in global coordinates, the future infinity \mathcal{I}^+ is 3d round sphere S^3 ,

$$ds_{\text{dS}}^2 = -dt^2 + \cosh^2(t) (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) \quad . \quad (2.2)$$

There are however two other forms of the metric in the coordinates which cover only a part of the space-time

$$ds_{\text{dS}}^2 = -dt^2 + \exp(2t) (d\chi^2 + \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2)) \quad (2.3)$$

²Below we call it “pure de Sitter space-time”.

and

$$ds_{\text{dS}}^2 = -dt^2 + \sinh^2(t) (d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) \quad . \quad (2.4)$$

In all three cases the metric $g_{ij}^{(0)}$ on asymptotic boundary is of constant curvature and it satisfies condition $R_{ij}^{(0)} = \frac{1}{3}g_{ij}^{(0)}R^{(0)}$. Notice that not all 4d metrics which approach 3d manifold of constant curvature at future infinity are globally identical to pure de Sitter space-time.

The expansion of the volume $V(\tau, t_q)$ in powers of e^{-t_q} is then a expansion in curvature of 3-dimensional Euclidean metric $g_{ij}^{(0)}$ defined on \mathcal{I}^+ at point q^+

$$V(\tau, t_q) = a_0(\tau) + a_2(\tau)R_{(0)}e^{-2t_q} + \left(c_4(\tau)\nabla^2 R_{(0)} + a_4(\tau)(R_{ij}^{(0)})^2 + b_4(\tau)R_{(0)}^2 \right) e^{-4t_q}, \quad (2.5)$$

where the curvature is taken at point q^+ . There are no combinations of curvature which are odd in derivatives. This explains why the coefficients in front of odd powers of e^{-t_q} in (2.5) vanish.

One can easily get some constraints on the coefficients in (2.5). If the spacetime is the pure de Sitter spacetime, then the volume of the causal diamond does not depend on where this diamond is located. This is a direct consequence of the large symmetry group in de Sitter space-time. It follows that the volume $V_{\text{dS}}(\tau, t_q)$ does not depend on t_q . The direct calculation performed for metric in any form (2.2), (2.3) or (2.4) gives

$$V_{\text{dS}}(\tau, t_q) = v(\tau) \equiv \frac{4}{3}\pi \left(2 \ln \cosh \frac{\tau}{2} - \tanh^2 \frac{\tau}{2} \right) \quad . \quad (2.6)$$

In this case all terms in the expansion (2.5) except the first should vanish. For metric (2.2) the spacelike infinity \mathcal{I}^+ is 3-sphere with curvature $R_{ij}^{(0)} = 2g_{ij}^{(0)}$, $R_{(0)} = 6$. Thus, we get that

$$a_0(\tau) = v(\tau), \quad a_2(\tau) = 0, \quad b_4(\tau) = -1/3a_4(\tau) \quad , \quad (2.7)$$

where $v(\tau)$ is the volume of the diamond in pure de Sitter spacetime. That coefficient $a_2(\tau)$ identically vanishes was checked explicitly in [2]. Thus we have that

$$V(\tau, t_q) = v(\tau) + \left(c_4(\tau)\nabla^2 R_{(0)} + a_4(\tau) \left((R_{ij}^{(0)})^2 - \frac{1}{3}R_{(0)}^2 \right) \right) e^{-4t_q} + \dots \quad . \quad (2.8)$$

The functions $a_4(\tau)$ and $c_4(\tau)$ can not be determined from general arguments and one has to perform a direct calculation.

We use in \mathcal{I}^+ the Riemann coordinates centered at point q^+ and directly compute the volume. The calculation shows that $c_4(\tau) = 0$ identically and that $a_4(\tau) = -w(\tau)$ is entirely negative function of τ ,

$$V(\tau, t_q) = v(\tau) - w(\tau) \left(R_{ij}^{(0)} - \frac{1}{3}g_{ij}^{(0)}R^{(0)} \right)^2 e^{-4t_q} + \dots \quad . \quad (2.9)$$

Thus, the cosmological evolution defines a volume flow of a causal diamond so that the volume is monotonically increasing function of cosmological time. The asymptotic value of

the volume is that of in pure de Sitter spacetime. Moreover, the flow vanishes (to leading order) if future infinity \mathcal{I}^+ is 3-manifold of constant curvature.

In the presence of 4d matter the monotonic behavior of the volume of a causal diamond persists although the deviations from the pure de Sitter result show up already in the second order in e^{-t_q} . This is also obvious from the analysis similar to (2.5): one can use the asymptotic values of the matter fields to construct new invariants that may appear in the expansion of the volume together with the curvature invariants. For a massless scalar field which takes value $\phi_0(x)$ on \mathcal{I}^+ the asymptotic expansion of the volume takes the form

$$V(\tau, t_q) = v(\tau) - (2\pi G_N)c(\tau)(\nabla\phi_0)^2 e^{-2t_q} + \dots \quad , \quad (2.10)$$

where G_N is 4d Newton's constant, $c(\tau)$ is positive function of τ and $(\nabla\phi_0)^2 = g_{(0)}^{ij}\partial_i\phi_0\partial_j\phi_0$.

Below we present a mathematical proof of (2.9) and (2.10).

3. Asymptotic metric and the Riemann coordinates

We choose a time coordinate $\eta = e^{-t}$, $\eta \geq 0$, $\eta = 0$ at future infinity. Note that we are using a convention in which η is positive and decreases towards future timelike infinity \mathcal{I}^+ .

The asymptotic expansion of cosmological 4-metric with positive cosmological constant was first considered by Starobinsky [7]. It goes similarly to the expansion in asymptotically anti-de Sitter case, see [8–10] for more detail on the anti-de Sitter case. The analytic continuation to the de Sitter case was considered in [11, 14] and [2]. The 4-dimensional metric takes the form

$$ds^2 = \frac{1}{\eta^2} (-d\eta^2 + g_{ij}(x, \eta)dx^i dx^j) \quad , \quad (3.1)$$

$$g(x, \eta) = g^{(0)}(x) + g^{(2)}(x)\eta^2 + g^{(3)}(x)\eta^3 + g^{(4)}(x)\eta^4 + \dots \quad ,$$

where $\{x^i\}$ are coordinates on \mathcal{I}^+ . The coefficients in the decomposition (3.1) satisfy relations [7]

$$g_{ij}^{(2)} = R_{ij}^{(0)} - \frac{1}{4}R^{(0)}g_{ij}^{(0)} \quad , \quad \text{Tr } g^{(3)} = 0 \quad , \quad \nabla^j g_{ij}^{(3)} = 0 \quad , \quad \text{Tr } g^{(4)} = \frac{1}{4}\text{Tr } g_{(2)}^2 \quad , \quad (3.2)$$

where the covariant derivatives and trace are determined with respect to metric $g_{ij}^{(0)}(x)$ defined on 3-surface \mathcal{I}^+ .

Now on the surface \mathcal{I}^+ we choose the Riemann coordinates $\{x^i\}$ such that $x^i = 0$ correspond to point q^+ . Locally, around point q^+ , it is more convenient to use the ‘‘spherical coordinates’’ (r, θ^a) on \mathcal{I}^+ ,

$$x^i = r n^i(\theta) \quad , \quad i = 1, 2, 3 \quad n^i(\theta)n^i(\theta) = 1 \quad , \quad (3.3)$$

where $\{\theta^a, a = 1, 2\}$ are the angle coordinates on S_2 . We then develop a double expansion of metric both in powers of η and $r^2 = x^i x^i$

$$g_{ij}^{(0)} = g_{ij}^{(0,0)} + g_{ij}^{(0,2)}r^2 + g_{ij}^{(0,3)}r^3 + g_{ij}^{(0,4)}r^4 + \dots \quad ,$$

$$g^{(2)} = g^{(2,0)} + g^{(2,1)}r + g^{(2,2)}r^2 + \dots \quad ,$$

$$g^{(3)} = g^{(3,0)} + g^{(3,1)}r + \dots \quad ,$$

$$g^{(4)} = g^{(4,0)} + \dots \quad , \quad (3.4)$$

where we keep terms up to 4th order in the total power of η and r . In the Riemann coordinates we have that [12]

$$\begin{aligned}
 g_{ij}^{(0,0)} &= \delta_{ij} \ , & (3.5) \\
 g_{ij}^{(0,2)} &= -\frac{1}{3}R_{ikjn}n^kn^n \ , \\
 g_{ij}^{(0,3)} &= -\frac{1}{6}R_{ikjn,l}n^kn^n n^l \ , \\
 g_{ij}^{(0,4)} &= \left(-\frac{1}{20}R_{ikjn,lm} + \frac{2}{45}R_{kin}{}^\rho R_{ljm\rho} \right) n^kn^n n^l n^m \ , \\
 g_{ij}^{(2,0)} &= (R_{ij} - \frac{1}{4}R\delta_{ij}) \ , \\
 g_{ij}^{(2,1)} &= \nabla_k(R_{ij} - \frac{1}{4}Rg_{ij})n^k \ , \\
 g_{ij}^{(2,2)} &= \left(\frac{1}{2}\nabla_k\nabla_n \left(R_{ij} - \frac{1}{4}Rg_{ij} \right) - \frac{1}{6}R^\rho{}_{kin} \left(R_{\rho j} - \frac{1}{4}Rg_{\rho j} \right) - \frac{1}{6}R^\rho{}_{kjn} \left(R_{\rho i} - \frac{1}{4}Rg_{\rho i} \right) \right) n^kn^n \ .
 \end{aligned}$$

Expanding (3.2) in powers of r we get the relations

$$\text{Tr } g^{(3,0)} = \text{Tr } g^{(3,1)} = 0, \quad \text{Tr } g^{(4,0)} = \frac{1}{4}\text{Tr } g_{(2,0)}^2 \ . \quad (3.6)$$

4. The future and past light-cones

We choose the point q to have coordinates $(\eta = \epsilon, 0, 0, 0)$ and point p to have coordinates $(\eta = N + \epsilon, 0, 0, 0)$, where $N = \epsilon(e^\tau - 1)$ and τ is the geodesic distance between points p and q , $\epsilon = e^{-tq}$.

In coordinates (η, r, θ^a) the equation which determines the past light-cone $\dot{I}^-(q)$, $r = r_+(\eta)$, $\theta^a = \text{const}$ is

$$\frac{dr_+(\eta)}{d\eta} = \frac{1}{\sqrt{g_{nn}}}, \quad g_{nn} = g_{ij}(x, \eta)n^i n^j, \quad r_+(\eta = \epsilon) = 0 \ . \quad (4.1)$$

We find that

$$\frac{1}{\sqrt{g_{nn}}} = 1 - \frac{\eta^2}{2}(g_{nn}^{(2,0)} + g_{nn}^{(2,1)}r + g_{nn}^{(2,2)}r^2) - \frac{\eta^3}{2}(g_{nn}^{(3,0)} + g_{nn}^{(3,1)}r) - \frac{\eta^4}{2}g_{nn}^{(4,0)} + \frac{3}{8}\eta^4(g_{nn}^{(2,0)})^2 + \dots$$

where we introduced notations $g_{nn}^{(k,p)} = g_{ij}^{(k,p)}n^i n^j$. The solution of equation (4.1) including the terms up to 5th order takes the form

$$\begin{aligned}
 r_+(\eta) &= \eta - \epsilon + \frac{\epsilon^3}{6}g_{nn}^{(2,0)} + \left(\frac{g_{nn}^{(3,0)}}{8} - \frac{g_{nn}^{(2,1)}}{24} \right) \epsilon^4 + \left(\frac{g_{nn}^{(2,2)}}{60} + \frac{g_{nn}^{(4,0)}}{10} - \frac{g_{nn}^{(3,1)}}{40} - \frac{3}{40}(g_{nn}^{(2,0)})^2 \right) \epsilon^5 \\
 &\quad - \frac{\eta^3}{6}(g_{nn}^{(2,0)} - \epsilon g_{nn}^{(2,1)} + \epsilon^2 g_{nn}^{(2,2)}) - \frac{\eta^4}{8}(g_{nn}^{(3,0)} + g_{nn}^{(2,1)} - \epsilon g_{nn}^{(3,1)} - 2\epsilon g_{nn}^{(2,2)}) \\
 &\quad - \frac{\eta^5}{10} \left(g_{nn}^{(2,2)} + g_{nn}^{(4,0)} + g_{nn}^{(3,1)} - \frac{3}{4}(g_{nn}^{(2,0)})^2 \right) + \dots \ . \quad (4.2)
 \end{aligned}$$

Similarly, the equation that determines the light-cone $\dot{I}^+(p)$, $r = r_-(\eta)$, $\theta^a = const$

$$\frac{dr_-(\eta)}{d\eta} = -\frac{1}{\sqrt{g_{nn}}}, \quad r_-(\eta = N + \epsilon) = 0 \quad . \quad (4.3)$$

Introducing $\bar{\epsilon} = N + \epsilon$ we have that

$$\begin{aligned} r_-(\eta) = & \bar{\epsilon} - \eta - \frac{\bar{\epsilon}^3}{6} g_{nn}^{(2,0)} - \left(\frac{g_{nn}^{(3,0)}}{8} + \frac{g_{nn}^{(2,1)}}{24} \right) \bar{\epsilon}^4 - \left(\frac{g_{nn}^{(2,2)}}{60} + \frac{g_{nn}^{(4,0)}}{10} + \frac{g_{nn}^{(3,1)}}{40} - \frac{3}{40} (g_{nn}^{(2,0)})^2 \right) \bar{\epsilon}^5 \\ & + \frac{\eta^3}{6} (g_{nn}^{(2,0)} + \bar{\epsilon} g_{nn}^{(2,1)} + \bar{\epsilon}^2 g_{nn}^{(2,2)}) + \frac{\eta^4}{8} (g_{nn}^{(3,0)} - g_{nn}^{(2,1)} + \bar{\epsilon} g_{nn}^{(3,1)} - 2\bar{\epsilon} g_{nn}^{(2,2)}) \\ & + \frac{\eta^5}{10} \left(g_{nn}^{(2,2)} + g_{nn}^{(4,0)} - g_{nn}^{(3,1)} - \frac{3}{4} (g_{nn}^{(2,0)})^2 \right) + \dots \quad . \end{aligned} \quad (4.4)$$

Two light-cones, $\dot{I}^+(p)$ and $\dot{I}^-(q)$, intersect at

$$\eta = \eta_c \equiv \frac{N}{2} + \epsilon - \frac{1}{8} g_{nn}^{(2,0)} N^2 \left(\frac{N}{2} + \epsilon \right) + O(\epsilon^4) \quad . \quad (4.5)$$

5. The volume

The volume of the causal diamond $\dot{I}^+(p) \cap \dot{I}^-(q)$ is

$$V(\epsilon, \tau) = \int_{S_2} \left(\int_{\epsilon}^{\eta_c} \frac{d\eta}{\eta^4} \int_0^{r_+(\eta)} dr r^2 \sqrt{\det g} + \int_{\eta_c}^{N+\epsilon} \frac{d\eta}{\eta^4} \int_0^{r_-(\eta)} dr r^2 \sqrt{\det g} \right) \quad , \quad (5.1)$$

where \int_{S_2} is the integral over spherical angles $\{\theta^a, a = 1, 2\}$. With the usual choice of angles $\int_{S_2} = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi$.

Then we get for the past light-cone of point q

$$\int_0^{r_+(\eta)} dr r^2 \sqrt{\det g} = \frac{1}{3} (\eta - \epsilon)^3 + S_5^+ + S_6^+ + S_7^+ \quad , \quad (5.2)$$

where the exact form of the coefficients S_5^+, S_6^+ is given in appendix A. The coefficient S_7^+ takes the form

$$S_7^+ = \sum_{n=0}^4 h_n^+ \eta^n (\epsilon - \eta)^{7-n} \quad , \quad (5.3)$$

where h_n^+ , $n = 0, 1, 2, 3, 4$ are presented in appendix A.

For the future light-cone of point p we have that

$$\int_0^{r_-(\eta)} dr r^2 \sqrt{\det g} = \frac{1}{3} (\bar{\epsilon} - \eta)^3 + S_5^- + S_6^- + S_7^- \quad . \quad (5.4)$$

The exact form of S_5^- and S_6^- can be found in appendix A. Using exact expressions of appendix A we notice a property

$$S_5^-(\bar{\epsilon}, \eta) = -S_5^+(\epsilon = \bar{\epsilon}, \eta) \quad . \quad (5.5)$$

For the term S_7^- we find

$$S_7^- = \sum_{n=0}^4 h_n^- \eta^n (\bar{\epsilon} - \eta)^{7-n} , \quad (5.6)$$

where h_n^- , $n = 0, 1, 2, 3, 4$ are given by relation (A.16). Using the S_2 integrals calculated in appendix B, we get for the integrated quantities

$$\begin{aligned} \int_{S_2} h_4^+ &= 0 , \\ \int_{S_2} h_3^+ &= -\frac{B}{3} + \frac{5C}{48} , \\ \int_{S_2} h_2^+ &= \frac{4C}{45} - \frac{B}{3} , \\ \int_{S_2} h_1^+ &= \frac{5C}{144} - \frac{B}{6} + \frac{A}{120} , \\ \int_{S_2} h_0^+ &= -\frac{113B}{3150} + \frac{227C}{37800} + \frac{A}{280} , \end{aligned} \quad (5.7)$$

where we introduced $A \equiv \pi \nabla^2 R$, $B \equiv \pi R_{ij}^2$, $C \equiv \pi R^2$.

We notice that since $\int_{S_2} g_{nn}^{(3,1)} = 0$ one has a relation

$$\int_{S_2} h_n^- = - \int_{S_2} h_n^+ , \quad n = 0, 1, 2, 3, 4 . \quad (5.8)$$

Now we are in a position to calculate the volume of the causal diamond. We focus on the term proportional to ϵ^4 since all other terms, proportional to ϵ^2 and ϵ^3 vanish. First, we neglect the modification (4.5) and assume that two light-cones intersect at $\eta_c = \frac{N}{2} + \epsilon$. We then get

$$V_1^{(4)} = \int_{\epsilon}^{\frac{N}{2} + \epsilon} \frac{d\eta}{\eta^4} S_7^+(\epsilon, \eta) + \int_{\frac{N}{2} + \epsilon}^{N + \epsilon} \frac{d\eta}{\eta^4} S_7^-(\bar{\epsilon}, \eta) = \sum_{n=0}^4 h_n^+ I_n(K) \epsilon^4 , \quad (5.9)$$

where

$$I_n(K) = \int_1^K dx (1-x)^{7-n} \left(\frac{1}{x^{4-n}} + \frac{1}{(x-2K)^{4-n}} \right) \quad (5.10)$$

and we introduced $K = \frac{N}{2\epsilon} + 1 = \frac{e^{\tau+1}}{2}$. There are useful relations between functions $I_n(K)$

$$\begin{aligned} I_2(K) &= -\frac{1}{3} I_1(K) - \frac{(1-K)^6}{3K^2} , \\ I_3(K) &= -\frac{1}{5} I_2(K) , \\ I_1(K) &= -\frac{3}{7} I_0(K) . \end{aligned} \quad (5.11)$$

Now we can take into account the modification (4.5) of the intersection point of the two light cones,

$$\eta_c = \frac{N}{2} + \epsilon + \Delta\eta_c, \quad \Delta\eta_c = -\frac{1}{2}g_{nn}^{(2,0)}K(K-1)^2\epsilon^3. \quad (5.12)$$

We get the contribution to the volume due to this modification

$$\begin{aligned} V_2^{(4)} &= \int_{S_2} \int_{\frac{N}{2}+\epsilon}^{\frac{N}{2}+\epsilon+\eta_c} \frac{d\eta}{\eta^4} \left(\frac{(\eta-\epsilon)^3}{3} - \frac{(\bar{\epsilon}-\eta)^3}{3} + \frac{1}{2}g_{nn}^{(2,0)}\eta[(\eta-\epsilon)^4 + (\bar{\epsilon}-\eta)^4] + \dots \right) \\ &= -\frac{1}{4} \int_{S_2} (g_{nn}^{(2,0)})^2 \frac{(K-1)^6}{K^2} \epsilon^4 + O(\epsilon^5). \end{aligned} \quad (5.13)$$

The total volume then

$$V^{(4)} = V_1^{(4)} + V_2^{(4)} = \epsilon^4 \left(\frac{2(K-1)^6}{135K^2} + \frac{4}{4725}I_0(K) \right) (C-3B). \quad (5.14)$$

Notice that terms proportional to $A = \nabla^2 R$ cancel each other. It is crucial for establishing the monotonic behavior of the volume because the term $\nabla^2 R$ is not sign-definite. On the other hand, one has that

$$(C-3B) = \pi(-3(R_{ij}^{(0)})^2 + R_{(0)}^2) = -3\pi \left(R_{ij}^{(0)} - \frac{1}{3}R^{(0)}g_{ij}^{(0)} \right)^2.$$

The integral $I_0(K)$ is calculated explicitly. Since $K = \frac{e^\tau+1}{2}$ one has that

$$\begin{aligned} \left(\frac{2(K-1)^6}{135K^2} + \frac{4}{4725}I_0(K) \right) &\equiv \frac{1}{3\pi}w(\tau) \\ &= \frac{2}{135} \left(1 - \tanh \frac{\tau}{2} \right)^{-4} \left(\frac{1}{15} \tanh^2 \frac{\tau}{2} \left(\tanh^4 \frac{\tau}{2} + 12 \tanh^2 \frac{\tau}{2} + \frac{6}{\cosh^4 \frac{\tau}{2}} + \frac{53}{\cosh^2 \frac{\tau}{2}} + 251 \right) \right. \\ &\quad \left. - 2 \left(1 + \tanh \frac{\tau}{2} \right)^4 \ln \left(1 + \tanh \frac{\tau}{2} \right) - 2 \left(1 - \tanh \frac{\tau}{2} \right)^4 \ln \left(1 - \tanh \frac{\tau}{2} \right) \right). \end{aligned} \quad (5.15)$$

The function $w(\tau)$ is positive and rapidly growing with τ . Finally, we get for the volume of the causal diamond

$$V(\tau, t_q) = v(\tau) - w(\tau) \left(R_{ij}^{(0)} - \frac{1}{3}R^{(0)}g_{ij}^{(0)} \right)^2 e^{-4t_q}, \quad (5.16)$$

where $v(\tau)$ is the volume in maximally symmetric de Sitter spacetime. Thus, we get that

$$\frac{dV(\tau, t_q)}{dt_q} = 4w(\tau) \left(R_{ij}^{(0)} - \frac{1}{3}R^{(0)}g_{ij}^{(0)} \right)^2 e^{-4t_q} > 0. \quad (5.17)$$

So that the volume of the causal diamond monotonically grows. The derivative (5.17) vanishes if curvature of \mathcal{I}^+ satisfies relation $(R_{ij}^{(0)} = \frac{1}{3}R^{(0)}g_{ij}^{(0)})$. By Bianchi identities this implies that $R^{(0)} = \text{const}$ so that \mathcal{I}^+ is 3-manifold of constant curvature in this case.

6. Coupling to massless scalar field

In the presence of matter fields the behavior (5.16) changes in that the correction term to the pure de Sitter result appears now at a different power of e^{-t} . In order to illustrate this point we consider a massless scalar field described by the field equation

$$\frac{1}{\sqrt{G}}\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu\phi) = 0 \quad , \quad (6.1)$$

where $G_{\mu\nu}$ is 4-metric. The Einstein equations take the form

$$R_{\mu\nu} = 3G_{\mu\nu} + 8\pi G_N\partial_\mu\phi\partial_\nu\phi \quad . \quad (6.2)$$

For the 4-metric in the form (3.1), introducing a coordinate $\rho = \eta^2$, the Einstein equations reduce to a system of equations [10]

$$\begin{aligned} \rho[2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g']_{ij} + R_{ij}(g) - (d-2)g'_{ij} - \text{Tr}(g^{-1}g')g_{ij} &= 8\pi G_N\partial_i\phi\partial_j\phi \quad , \\ \nabla_i \text{Tr}(g^{-1}g') - \nabla^j g'_{ij} &= 8\pi G_N\partial_\rho\phi\partial_i\phi \quad , \\ \text{Tr}(g^{-1}g'') - \frac{1}{2}\text{Tr}(g^{-1}g'g^{-1}g') &= 16\pi G_N\partial_\rho\phi\partial_\rho\phi \quad , \end{aligned} \quad (6.3)$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g , and $R_{ij}(g)$ is the Ricci tensor of g .

The asymptotic expansion for the scalar field and the 4-metric reads

$$\phi(\rho, x) = \phi^{(0)}(x) + \phi^{(2)}(x)\rho + \dots \quad , \quad (6.4)$$

$$g_{ij}(\rho, x) = g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x)\rho + \dots \quad . \quad (6.5)$$

Inserting this into the first equation in (6.3) we find that

$$g_{ij}^{(2)} = \left(R_{ij}^{(0)} - \frac{1}{4}g_{ij}^{(0)}R^{(0)} \right) - 8\pi G_N \left(\partial_i\phi^{(0)}\partial_j\phi^{(0)} - \frac{1}{4}g_{(0)}^{ij}\partial_i\phi^{(0)}\partial_j\phi^{(0)} \right) \quad (6.6)$$

and, hence, for the trace

$$\text{Tr}g^{(2)} = \frac{1}{4}R^{(0)} - 2\pi G_N g_{(0)}^{ij}\partial_i\phi^{(0)}\partial_j\phi^{(0)} \quad . \quad (6.7)$$

Before computing the volume of the causal diamond we notice that all our expressions obtained in the previous section and in appendix A and that operate with coefficients in the expansion (3.4) and do not refer to the precise form of the coefficients are also valid in the case when the bulk gravity couples to matter. We also note that regardless to the precise form of the coefficient $g_{ij}^{(2)}$ we have that

$$\int_{S_2} g_{nn}^{(2,0)} = \frac{1}{3} \int_{S_2} \text{Tr}g^{(2,0)} \quad . \quad (6.8)$$

The volume can be computed in the same way as in the previous section. The only difference is that now the term of order ϵ^2 does not identically vanish. We get that

$$V(\tau, t_q) = v(\tau) + \int_{S_2} \left(\int_\epsilon^{\frac{N}{2}+\epsilon} \frac{d\eta}{\eta^4} S_5^+(\epsilon, \eta) + \int_{\frac{N}{2}+\epsilon}^{N+\epsilon} \frac{d\eta}{\eta^4} S_5^-(\bar{\epsilon}, \eta) \right) + \dots \quad . \quad (6.9)$$

Inserting here the exact expressions for S_5^+ and S_5^- we arrive at the expression

$$V(\tau, \epsilon) = v(\tau) + \left(\frac{1}{2} \int_{S_2} g_{nn}^{(2,0)} J_1(\tau) + \left(\frac{1}{10} \int_{S_2} \text{Tr} g^{(0,2)} - \frac{1}{6} \int_{S_2} g_{nn}^{(2,0)} \right) J_2(\tau) \right) \epsilon^2 + \dots \quad (6.10)$$

where we introduced functions

$$J_1(\tau) = \int_1^K dx (x-1)^4 \left(\frac{1}{x^3} + \frac{1}{(x-2K)^3} \right) \quad (6.11)$$

and

$$J_2(\tau) = \int_1^K dx (x-1)^5 \left(\frac{1}{x^4} + \frac{1}{(x-2K)^4} \right) \quad (6.12)$$

where $K = \frac{e^\tau + 1}{2}$. We note a relation between two functions

$$J_2(\tau) = \frac{5}{3} J_1(\tau) \quad (6.13)$$

Taking into account that $g_{ij}^{(0,2)} = -\frac{1}{3} R_{ikjn} n^k n^n$ we find

$$\begin{aligned} \int_{S_2} \text{Tr} g^{(0,2)} &= -\frac{4\pi}{9} R_{(0)} \quad , \\ \int_{S_2} g_{nn}^{(2,0)} &= \frac{4\pi}{3} \left(\frac{1}{4} R_{(0)} - 2\pi G_N (\nabla_{(0)} \phi_{(0)})^2 \right) \quad , \end{aligned} \quad (6.14)$$

where all quantities are calculated at point q^+ on \mathcal{I}^+ .

Thus, we obtain for the volume

$$V(\tau, t_q) = v(\tau) - \frac{8\pi}{27} J_1(\tau) (2\pi G_N) (\nabla_{(0)} \phi_{(0)})^2 e^{-2t_q} + \dots \quad (6.15)$$

The function $J_1(\tau)$ is positive and rapidly growing with τ ,

$$J_1(\tau) = \frac{\tau^6}{320} + \frac{\tau^7}{320} + \frac{19}{17920} \tau^8 + O(\tau^9) \quad , \quad \tau \ll 1 \quad (6.16)$$

$$J_1(\tau) = \left(\frac{17}{4} - 6 \ln 2 \right) e^{2\tau} - \frac{3}{2} e^\tau + O(1) \quad , \quad \tau \gg 1 \quad (6.17)$$

Clearly, the volume (6.15) is monotonically growing. It reaches asymptotically the maximal value that is the volume of the diamond in pure de Sitter space-time. The volume flow (6.15) vanishes everywhere in \mathcal{I}^+ if and only if the scalar field takes a constant asymptotic value $\phi_{(0)}$ on \mathcal{I}^+ .

7. Two conjectures

It is reasonable to ask whether the volume of a causal diamond is globally monotonic or, rather, it is monotonic only asymptotically. It is known [13] (see also [14]) that there exists a small perturbation of pure de Sitter space-time that does not change the global structure

of the space-time. It makes a small deformation of past and future infinities \mathcal{I}^- and \mathcal{I}^+ . Near the past infinity our analysis is valid by replacing t by $-t$. So that the volume is monotonically decreasing near \mathcal{I}^- . Thus in a space-time which is globally asymptotically de Sitter, to the past and future, \mathcal{I}^- and \mathcal{I}^+ , the volume of a causal diamond is *not* globally monotonic. On the other hand, a positive energy distribution generically causes a formation of a initial singularity. The space-time near the past infinity is then no more asymptotically de Sitter. We expect that in space-times of this type the volume of the diamond is globally monotonic as a manifestation of the irreversibility of the cosmological evolution with a positive cosmological constant. A detail analysis, however, is yet to be done.

Irrespective of whether or not the volume is globally monotonic it is reasonable to expect that the volume of a diamond in pure de Sitter space-time is the absolute maximum among all possible asymptotically de Sitter space-times. Thus we formulate two conjectures:

1. *The volume of a causal diamond in pure de Sitter space-time is the absolute maximum in the class of vacuum asymptotically de Sitter metrics*

$$V(\tau, t_q) \leq V_{\text{dS}}(\tau) \quad . \quad (7.1)$$

2. *The bound (7.1) is saturated for all diamonds of same duration τ if and only if the space-time is pure de Sitter space-time.*

A more work is needed to check these statements.

It should be noted that the assumption of an asymptotically de Sitter space-time at $t \rightarrow \infty$ is crucial for all results obtained in the paper. The only known generic alternative behaviour is recollapse with the subsequent formation of a singularity. We do not expect the volume of a diamond to be monotonic in this scenario although a detail analysis is needed in order to specify the time evolution of the volume.

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A. Volume coefficients

Using the decomposition

$$\begin{aligned} \sqrt{\det g} &= 1 + \frac{r^2}{2} \text{Tr} g^{(0,2)} + \frac{r^3}{2} \text{Tr} g^{(0,3)} + \frac{r^4}{2} \text{Tr} g^{(0,4)} + \frac{1}{2} \eta^2 (\text{Tr} g^{(2,0)} + r \text{Tr} g^{(2,1)} + r^2 \text{Tr} g^{(2,2)}) \\ &\quad + \frac{1}{8} (r^2 \text{Tr} g^{(0,2)} + \eta^2 \text{Tr} g^{(2,0)})^2 - \frac{r^4}{4} \text{Tr} (g^{(0,2)} g^{(0,2)}) \\ &\quad - \frac{r^2 \eta^2}{2} \text{Tr} (g^{(0,2)} g^{(2,0)}) - \frac{\eta^4}{8} \text{Tr} (g^{(2,0)} g^{(2,0)}) + \dots \end{aligned} \quad (A.1)$$

we get

$$\int_0^{r_+(\eta)} dr r^2 \sqrt{\det g} = \frac{1}{3} (\eta - \epsilon)^3 + S_5^+ + S_6^+ + S_7^+ \quad , \quad (A.2)$$

where

$$S_5^+ = \frac{1}{6}(\text{Tr } g^{(2,0)} - 3g_{nn}^{(2,0)})\eta^2(\eta - \epsilon)^3 + \frac{1}{2}g_{nn}^{(2,0)}\eta(\eta - \epsilon)^4 \quad (\text{A.3})$$

$$+ \left(\frac{1}{10}\text{Tr } g^{(0,2)} - \frac{1}{6}g_{nn}^{(2,0)} \right) (\eta - \epsilon)^5$$

$$S_6^+ = -\frac{g_{nn}^{(3,0)}}{2}\epsilon^3(\eta - \epsilon)^3 + \left(-\frac{g_{nn}^{(2,1)}}{4} - \frac{3}{4}g_{nn}^{(3,0)} + \frac{\text{Tr } g^{(2,1)}}{8} \right) \epsilon^2(\eta - \epsilon)^4 \quad (\text{A.4})$$

$$+ \left(-\frac{g_{nn}^{(2,1)}}{3} - \frac{g_{nn}^{(3,0)}}{2} + \frac{\text{Tr } g^{(2,1)}}{4} \right) \epsilon(\eta - \epsilon)^5$$

$$+ \left(\frac{\text{Tr } g^{(2,1)}}{8} + \frac{\text{Tr } g^{(0,3)}}{12} - \frac{g_{nn}^{(3,0)}}{8} - \frac{g_{nn}^{(2,1)}}{8} \right) (\eta - \epsilon)^6$$

For the coefficient S_7^+ we get a representation

$$S_7^+ = \sum_{n=0}^4 h_n^+ \eta^n (\epsilon - \eta)^{7-n} \quad (\text{A.5})$$

$$h_4^+ = \frac{1}{2}g_{nn}^{(4,0)} - \frac{5}{8}(g_{nn}^{(2,0)})^2 - \frac{1}{24}(\text{Tr } g^{(2,0)})^2 + \frac{1}{4}g_{nn}^{(2,0)}\text{Tr } g^{(2,0)} + \frac{1}{24}\text{Tr } (g^{(2,0)}g^{(2,0)}) \quad (\text{A.6})$$

$$h_3^+ = -\frac{5}{4}(g_{nn}^{(2,0)})^2 + \frac{1}{4}g_{nn}^{(2,0)}\text{Tr } g^{(2,0)} + g_{nn}^{(4,0)} - \frac{1}{4}g_{nn}^{(3,1)} \quad (\text{A.7})$$

$$h_2^+ = \frac{1}{4}g_{nn}^{(2,0)}\text{Tr } g^{(0,2)} - \frac{7}{6}(g_{nn}^{(2,0)})^2 - \frac{1}{10}\text{Tr } g^{(2,2)} + \frac{1}{10}\text{Tr } (g^{(0,2)}g^{(2,0)}) - \frac{1}{20}\text{Tr } g^{(2,0)}\text{Tr } g^{(0,2)}$$

$$+ \frac{1}{6}g_{nn}^{(2,2)} + g_{nn}^{(4,0)} + \frac{1}{12}g_{nn}^{(2,0)}\text{Tr } g^{(2,0)} - \frac{1}{4}g_{nn}^{(3,1)} \quad (\text{A.8})$$

$$h_1^+ = \frac{1}{4}g_{nn}^{(2,0)}\text{Tr } g^{(0,2)} - \frac{13}{24}(g_{nn}^{(2,0)})^2 + \frac{1}{12}g_{nn}^{(2,2)} + \frac{1}{2}g_{nn}^{(4,0)} - \frac{1}{8}g_{nn}^{(3,1)} \quad (\text{A.9})$$

$$h_0^+ = -\frac{37}{360}(g_{nn}^{(2,0)})^2 - \frac{1}{56}(\text{Tr } g^{(0,2)})^2 + \frac{1}{28}\text{Tr } (g^{(0,2)}g^{(0,2)})$$

$$- \frac{1}{14}\text{Tr } g^{(0,4)} + \frac{1}{60}g_{nn}^{(2,2)} + \frac{g_{nn}^{(4,0)}}{10} + \frac{1}{12}g_{nn}^{(2,0)}\text{Tr } g^{(0,2)} - \frac{1}{40}g_{nn}^{(3,1)} \quad (\text{A.10})$$

For the future cone of point p we have that

$$\int_0^{r-(\eta)} dr r^2 \sqrt{\det g} = \frac{1}{3}(\bar{\epsilon} - \eta)^3 + S_5^- + S_6^- + S_7^- \quad (\text{A.11})$$

$$S_5^- = -\frac{1}{6}(\text{Tr } g^{(2,0)} - 3g_{nn}^{(2,0)})\eta^2(\eta - \bar{\epsilon})^3 - \frac{1}{2}g_{nn}^{(2,0)}\eta(\eta - \bar{\epsilon})^4 \quad (\text{A.12})$$

$$- \left(\frac{1}{10}\text{Tr } g^{(0,2)} - \frac{1}{6}g_{nn}^{(2,0)} \right) (\eta - \bar{\epsilon})^5$$

$$S_6^- = \frac{g_{nn}^{(3,0)}}{2}\bar{\epsilon}^3(\eta - \bar{\epsilon})^3 + \left(-\frac{g_{nn}^{(2,1)}}{4} + \frac{3}{4}g_{nn}^{(3,0)} + \frac{\text{Tr } g^{(2,1)}}{8} \right) \bar{\epsilon}^2(\eta - \bar{\epsilon})^4 \quad (\text{A.13})$$

$$+ \left(-\frac{g_{nn}^{(2,1)}}{3} + \frac{g_{nn}^{(3,0)}}{2} + \frac{\text{Tr } g^{(2,1)}}{4} \right) \bar{\epsilon}(\eta - \bar{\epsilon})^5$$

$$+ \left(\frac{\text{Tr } g^{(2,1)}}{8} + \frac{\text{Tr } g^{(0,3)}}{12} + \frac{g_{nn}^{(3,0)}}{8} - \frac{g_{nn}^{(2,1)}}{8} \right) (\eta - \bar{\epsilon})^6$$

Clearly, we have a property

$$S_5^-(\bar{\epsilon}, \eta) = -S_5^+(\epsilon = \bar{\epsilon}, \eta) \quad . \quad (\text{A.14})$$

For the term S_7^- we find

$$S_7^- = \sum_{n=0}^4 h_n^- \eta^n (\bar{\epsilon} - \eta)^{7-n} \quad , \quad (\text{A.15})$$

where we have a relation

$$\begin{aligned} h_4^- &= -h_4^+ \quad , \\ h_3^- &= -h_3^+ - \frac{1}{2}g_{nn}^{(3,1)} \quad , \\ h_2^- &= -h_2^+ - \frac{1}{2}g_{nn}^{(3,1)} \quad , \\ h_1^- &= -h_1^+ - \frac{1}{4}g_{nn}^{(3,1)} \quad , \\ h_0^- &= -h_0^+ - \frac{1}{20}g_{nn}^{(3,1)} \quad . \end{aligned} \quad (\text{A.16})$$

Since $\int_{S_2} g_{nn}^{(3,1)} = 0$ we have that

$$\int_{S_2} h_n^- = - \int_{S_2} h_n^+ \quad , \quad n = 0, 1, 2, 3, 4 \quad (\text{A.17})$$

B. Spherical integrals

Calculating integrals over spherical angle coordinates we use that

$$\begin{aligned} \int_{S_2} n^i n^j &= \frac{4}{3} \pi \delta^{ij} \quad , \\ \int_{S_2} n^i n^j n^k n^l &= \frac{4}{15} \pi (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \quad . \end{aligned} \quad (\text{B.1})$$

We use the fact that in three dimensions

$$R_{ijkn} = g_{ik} P_{jn} + g_{jn} P_{ik} - g_{jk} P_{in} - g_{in} P_{jk} \quad , \quad P_{ij} = R_{ij} - \frac{1}{4} g_{ij} R \quad . \quad (\text{B.2})$$

We then get, introducing $A \equiv \pi \nabla^2 R$, $B \equiv \pi R_{ij}^2$, $C \equiv \pi R^2$,

$$\begin{aligned} \int_{S^2} g_{nn}^{(3,1)} &= 0 \quad , \\ \int_{S_2} g_{nn}^{(2,2)} &= \frac{A}{10} \quad , \\ \int_{S_2} g_{nn}^{(4,0)} &= \frac{B}{3} - \frac{5C}{48} \quad , \\ \int_{S_2} (g_{nn}^{(2,0)})^2 &= \frac{8B}{15} - \frac{3C}{20} \quad , \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned}
\int_{S_2} g_{nn}^{(2,0)} \text{Tr} g^{(2,0)} &= \frac{C}{12} , \\
\int_{S_2} g_{nn}^{(2,0)} \text{Tr} g^{(0,2)} &= \frac{C}{45} - \frac{8B}{45} , \\
\int_{S_2} (\text{Tr} g^{(2,0)})^2 &= \frac{C}{4} , \\
\int_{S_2} \text{Tr} g^{(2,0)} \text{Tr} g^{(0,2)} &= -\frac{C}{9} , \\
\int_{S_2} \text{Tr} g^{(0,2)} \text{Tr} g^{(0,2)} &= \frac{4C}{135} + \frac{8B}{135} , \\
\int_{S_2} \text{Tr} g^{(2,2)} &= \frac{A}{6} - \frac{4B}{9} + \frac{C}{9} , \\
\int_{S_2} \text{Tr} g^{(0,4)} &= -\frac{2A}{75} + \frac{56B}{675} - \frac{4C}{225} , \\
\int_{S_2} \text{Tr} (g^{(0,2)} g^{(2,0)}) &= -\frac{4B}{9} + \frac{C}{9} , \\
\int_{S_2} \text{Tr} (g^{(2,0)} g^{(2,0)}) &= 4B - \frac{5C}{4} , \\
\int_{S_2} \text{Tr} (g^{(0,2)} g^{(0,2)}) &= \frac{28B}{135} - \frac{2C}{45} .
\end{aligned}$$

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